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Asymptotic Behavior in a Second Order Linear Matrix Differential Equation

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0. INTRODUCTION

The purpose of this paper is to show that results similar to those of Hille [1] may be extended to the matrix case. Hille showed that for the scalar equation

$$y'' - F(t)\,y = 0 \tag{0.1}$$

where F has constant sign and $F \in L(\epsilon, \epsilon^{-1})$ for all $\epsilon \in (0,1)$, there is a solution $y(t)$ of (0.1) satisfying $y(\infty) = 1$, if and only if $tF(t) \in L(1, \infty)$. Here we will be studying the matrix equation

$$X'' = A(t)\,X \tag{0.2}$$

on the interval $[a, \infty)$, and the terminal conditions

$$X(\infty) = I, \tag{0.3}$$

and

$$X(\infty) = \Lambda, \tag{0.4}$$

where Λ, A, X are $n \times n$ matrices, I is the identity, and A is continuous on $[a, \infty)$. For convenience we assume $a \geqslant 0$. Further, we will consider the effect of the eigenvalues of $A(t)$ on the terminal matrix Λ. In keeping with the assumption of the constant sign of F in (0.1), for most of the results we will assume $A(t)$ is positive semidefinite or negative semidefinite denoted respectively by $A(t) \geqslant 0$ and $A(t) \leqslant 0$. By this we will mean that $A(t)$ is symmetric, and that the quadratic form $c^*A(t)c$ satisfies $c^*A(t)c \geqslant 0$ and $c^*A(t)c \leqslant 0$ respectively for each n-vector c. In either case $A(t)$ will be called semidefinite. It may be noted that some of the following results are true with obvious modifications when the symmetry condition is removed from the semidefiniteness assumption on $A(t)$.

For a symmetric matrix we note that its eigenvalues are real and, hence, may be ordered. In this paper the eigenvalues of a symmetric matrix Q are written in order as

$$\lambda_1(Q) \leqslant \lambda_2(Q) \leqslant \cdots \leqslant \lambda_n(Q). \tag{0.5}$$

Further, Q^* denotes the transpose of the real matrix Q and

$$\| Q \| = \sup\{\| Qx \| : \| x \| = 1\}$$

where $\| x \|$ is the Euclidean norm of the n-vector x. We also note that when $X(t)$ is a solution to (0.2), and $A(t)$ is symmetric that

$$X'(t)^* X(t) - X^*(t) X'(t) = C,$$

a constant matrix. This is a consequence of the fact that the derivative of this expression is zero. When this constant C is the zero matrix we will call the solution $X(t)$ of (0.2) prepared, as Hartman did in [2].

In [3] Wintner showed that any bounded column vector solution $x(t)$ of (0.2), for $A(t) \geqslant 0$, satisfies $x'(t) \to 0$ as $t \to \infty$, even, in fact, if the symmetry condition on $A(t)$ is dropped. Hence, if $X(t)$ is a matrix solution of the problem (0.2) and (0.4), we see that $X'(t) \to 0$ as $t \to \infty$. Therefore, such a solution is prepared, as can be seen by letting $t \to \infty$ in the expression

$$C = X'(t)^* X(t) - X^*(t) X'(t).$$

1. Extension to $n \times n$ Matrices

Following Hille [1] we first note the following. (See also Barrett [8; Theorems 1.1 and 1.3].)

Lemma 1.1. *If $X(t)$ is a solution of the problem* (0.2) *and* (0.3) *where $A(t)$ is symmetric then the general solution for* (0.2) *is of the form*

$$[I + E(t)]\, C + t[I + G(t)]\, D \tag{1.1}$$

where C and D are constant matrices and $E(t)$ and $G(t)$ both converge to the zero matrix as $t \to \infty$.

Proof. Let $X(t) = I + E(t)$ where $E(t) \to 0$ as $t \to \infty$. Define

$$Y(t) = X(t) \int_b^t X^{-1}(s)\, X^{-1}(s)^*\, ds \tag{1.2}$$

where b is chosen large enough so that $X^{-1}(s)$ exists on $[b, \infty)$. Noting the nature of $X(t) = I + E(t)$, we see that $Y(t)$ has the form

$$Y(t) = t[I + G(t)] \tag{1.3}$$

where $G(t) \to 0$ as $t \to \infty$. However, $Y(t)$ as defined in (1.2) is a solution of (0.2) when $X(t)$ is a nonsingular prepared solution. Further, $Y(t)$ is independent of $X(t)$. Hence $X(t)\,C + Y(t)\,D$ is the general solution of (0.2).

Solutions may be attained by iteration just as in the scalar case in [1].

LEMMA 1.2. *If* $\int_a^\infty s \,\|A(s)\|\, ds < \infty$, *then there is a solution to the problem* (0.2) *and* (0.3).

Proof. Let $X_0(t) \equiv I$,

$$X_n(t) = I + \int_t^\infty (s - t)\, A(s)\, X_{n-1}(s)\, ds$$

and

$$g(t) = \int_t^\infty s \,\|A(s)\|\, ds.$$

We first show that

$$\| X_n(t) - X_{n-1}(t)\| \leqslant g(t)^n\, (n!)^{-1}.$$

Proceeding by induction,

$$\begin{aligned} \| X_1(t) - X_0(t)\| &= \left\| \int_t^\infty (s - t)\, A(s)\, ds \right\| \\ &\leqslant \int_t^\infty s \,\|A(s)\|\, ds \\ &= g(t), \qquad \text{for} \quad n = 1. \end{aligned}$$

Assume

$$\| X_k(t) - X_{k-1}(t)\| \leqslant g(t)^k\, (k!)^{-1}.$$

Then

$$\begin{aligned} \| X_{k+1}(t) - X_k(t)\| &= \left\| \int_t^\infty (s - t)\, A(s)[X_k(s) - X_{k-1}(s)]\, ds \right\| \\ &\leqslant \int_t^\infty s \,\|A(s)\|\, g(s)^k\, (k!)^{-1}\, ds \\ &= \int_t^\infty [-g'(s)]\, g(s)^k\, (k!)^{-1}\, ds = g(t)^{k+1}\, [(k + 1)!]^{-1}. \end{aligned}$$

Thus $X_n(t)$ converges to a matrix, say $X(t)$, which satisfies

$$X(t) = I + \int_t^\infty (s - t)\, A(s)\, X(s)\, ds. \tag{1.4}$$

Hence, we infer that (0.2) and (0.3) are both satisfied.

We also note in passing that $tX'(t) \to 0$ as $t \to \infty$ by virtue of the alternate formula

$$X(t) = X(b) + X'(b)(t - b) + \int_t^b (s - t)\, A(s)\, X(s)\, ds$$

which converges to

$$X(t) = I + \lim_{b\to\infty}[X'(b)(t - b)] + \int_t^\infty (s - t)\, A(s)\, X(s)\, ds \quad \text{as} \quad b \to \infty.$$

Comparing this with (1.4), we conclude that $bX'(b) \to 0$ as $b \to \infty$.

From [4, p. 181] we see that $\lambda_n[A(t)] = \| A(t)\|$ for $A(t) \geqslant 0$ and $-\lambda_1[A(t)] = \| A(t)\|$ for $A(t) \leqslant 0$. Hence the following is true.

COROLLARY 1.3. *If $A(t)$ is positive (negative) semidefinite, and if*

$$\int_a^\infty t\lambda_n[A(t)]\, dt < \infty \qquad \left(\int_a^\infty t\lambda_1[A(t)]\, dt > -\infty\right)$$

then there is a matrix solution $X(t)$ to the problem (0.2) *and* (0.3).

Next we state a partial converse to Lemma 1.2 noting that in the proof the semidefiniteness of $A(t)$ is needed.

THEOREM 1.4. *If $A(t)$ is semidefinite and if the problem* (0.2) *and* (0.3) *has a solution $U(t)$, then*

$$\int_a^\infty s \,\| A(s)\|\, ds < \infty. \tag{1.5}$$

Proof. We assume $A(t) \geqslant 0$. With certain obvious modifications, the following proof is also valid for $A(t) \leqslant 0$. Let $Y(t)$ be the second, independent solution given by formula (1.2) in Lemma 1.1. Then

$$Y'(t) = X^{-1}(t)^* + X'(t) \int_b^t X^{-1}(s)\, X^{-1}(s)^*\, ds$$

and $Y'(t) \to I$ as $t \to \infty$, since $X^{-1}(t)^* \to I$ and $tX'(t) \to 0$ as $t \to \infty$. Noting that

$$\begin{aligned} Y'(c) - Y'(b) &= \int_b^c Y''(s)\, ds \\ &= \int_b^c A(s)\, Y(s)\, ds \end{aligned}$$

and letting $c \to \infty$, we conclude that

$$\int_b^\infty A(s)\, Y(s) = I - Y'(b).$$

Therefore,

$$\left\| \int_b^\infty A(s)\, Y(s)\, ds \right\| < \infty.$$

Hence all the eigenvalues satisfy

$$\left| \lambda_k \left[\int_b^\infty A(s)\, Y(s)\, ds \right] \right| < \infty \qquad \text{for} \quad 1 \leqslant k \leqslant n,$$

though in general they are complex numbers and are not ordered as in the case of a symmetric matrix. Thus

$$-\infty < \operatorname{tr} \left[\int_b^\infty A(s)\, Y(s)\, ds \right] < +\infty$$

where $\operatorname{tr} Q$ denotes the trace of matrix Q. Further

$$-\infty < \int_b^\infty \operatorname{tr}[A(s)\, Y(s)]\, ds < \infty.$$

Using (1.3) as in Lemma 1.1, and $\operatorname{tr}(P + Q) = \operatorname{tr} P + \operatorname{tr} Q$, we have

$$-\infty < \int_a^\infty (\operatorname{tr}[sA(s)] + \operatorname{tr}[sA(s)\, G(s)])\, ds < \infty. \tag{1.6}$$

Let ϵ be arbitrary such that $0 < \epsilon < 1/n$. Since $G(t) \to 0$ as $t \to \infty$ there is $c \in [b, \infty)$ such that $| g_{ij}(s)| < \epsilon$ for $s \geqslant c$ and all $1 \leqslant i, j \leqslant n$, where $G = (g_{ij})$. Hence for $s \geqslant c$,

$$\begin{aligned} |\operatorname{tr}[sA(s)\, G(s)]| &\leqslant s \sum_{i=1}^n \sum_{j=1}^n | a_{ij}(s)|\, \epsilon \\ &\leqslant s\epsilon \sum_i \sum_j 2^{-1}[a_{ii}(s) + a_{jj}(s)] \\ &\leqslant n\epsilon s \operatorname{tr}[A(s)] < \operatorname{tr}[sA(s)] \quad \text{where} \quad A = (a_{ij}). \end{aligned}$$

To see that the second stage of this inequality is true, we note that

$$(e_i + e_j)^*\, A(s)(e_i + e_j) \geqslant 0, \qquad \text{and} \qquad (e_i - e_j)^*\, A(s)(e_i - e_j) \geqslant 0$$

where e_k is the unit vector with a one in the kth position and zeros elsewhere. These relations imply that $-(a_{ii} + a_{jj}) \leqslant 2a_{ij} \leqslant (a_{ii} + a_{jj})$. But, the above inequalities imply that

$$-\infty < -\int_c^\infty (n\epsilon \operatorname{tr}[sA(s)] + \operatorname{tr}[sA(s)\, G(s)])\, ds \leqslant 0. \tag{1.7}$$

Combining (1.6) and (1.7), we obtain

$$-\infty < \int_c^\infty (1 - n\epsilon) \operatorname{tr}[sA(s)]\, ds < \infty.$$

Therefore, since $A(s) \geqslant 0$, and $1 - n\epsilon > 0$, we have

$$0 \leqslant \int_c^\infty s \operatorname{tr} A(s)\, ds < \infty$$

which further implies that

$$0 \leqslant \int_c^\infty s\lambda_n[A(s)]\, ds < \infty.$$

But this gives us (1.5) as claimed.

At this point we can state the following.

COROLLARY 1.5. *Let $A(t) \geqslant 0$ ($A(t) \leqslant 0$). Then the problem* (0.2) *and* (0.3) *has a solution if and only if*

$$\int_a^\infty s\lambda_n[A(s)]\, ds < \infty \qquad \left(\int_a^\infty s\lambda_1[A(s)]\, ds > -\infty\right).$$

2. RELATIONSHIPS BETWEEN $\lambda_k[A(t)]$ AND Λ

Next we note how the behavior of the eigenvalues affects the terminal matrix.

THEOREM 2.1. *Let $A(t) \geqslant 0$, suppose $\int_a^\infty s\lambda_1[A(s)]\, ds = \infty$, and let $X(t)$ be a solution to the problem* (0.2) *and* (0.4). *Then $\Lambda = 0$.*

Proof. By way of contradiction assume $\Lambda \neq 0$. Then there is an n-vector c such that $c^*\Lambda^*\Lambda c \neq 0$. Using this vector c, define $x(t) = c^*X^*(t)\, X(t)\, c$. Since $Q^*Q \geqslant 0$ for all matrices Q, we have

$$x(t) \geqslant 0 \tag{2.1}$$

and

$$x(\infty) = c^*\Lambda^*\Lambda c \equiv \mu > 0. \tag{2.2}$$

By differentiation and the symmetry of $A(t)$, we see that

$$x'(t) = c^*[X^*(t)\, X'(t) + X'(t)^*\, X(t)]\, c,$$

and

$$x''(t) = 2c^*[X^*(t)\, A(t)\, X(t) + X'(t)^* X'(t)]\, c.$$

Thus

$$x''(t) \geqslant 0 \tag{2.3}$$

and

$$x''(t) \geqslant 2c^* X^*(t)\, A(t)\, X(t)\, c \geqslant 2\lambda_1[A(t)]\, x(t) \tag{2.4}$$

since $A(t) \geqslant 0$. Hence $x(t)$ is a lower solution (see [5]) of the scalar equation

$$y'' = 2\lambda_1[A(t)]\, y, \tag{2.5}$$

since $\lambda_1[A(t)] \geqslant 0$. We also note that $s(t) \equiv x(a)$ is an upper solution of (2.5). By virtue of (2.1), (2.2), and (2.3) we see that $x(t)$ is a positive, convex, nonincreasing function, and, therefore, that $0 < \mu \leqslant x(t) \leqslant s(t)$. Thus, by [6, Cor. 4.1] there is a solution $y(t)$ of (2.5) such that

$$0 < \mu \leqslant x(t) \leqslant y(t) \leqslant s(t).$$

This implies that

$$y''(t) = 2\lambda_1[A(t)]\, y(t) \geqslant 0,$$

and, therefore, that $y(t)$ also is a positive, convex, nonincreasing function. Hence, $y(\infty) = \gamma$ exists and $\gamma \geqslant \mu > 0$. Therefore, $\gamma^{-1}y(t)$ is a solution to (0.2) and (0.4) for $n = 1$ with $A(t)$ replaced by $2\lambda_1[A(t)]$. Finally, by the scalar case of Corollary 1.5,

$$\int_a^\infty 2t\lambda_1[A(t)]\, dt < \infty,$$

which contradicts the hypothesis, and the theorem is proven.

The reader may suspect that if

$$\int_a^\infty t\lambda_k[A(t)]\, dt < \infty$$

and

$$\int_a^\infty t\lambda_{k+1}[A(t)]\, dt = \infty$$

for $A(t) \geqslant 0$ that there will be a solution to the problem (0.2) and (0.4) with rank $A = k$. This, however, is not in general true as the following example indicates.

EXAMPLE 2.2. Let $A(t) = \operatorname{diag}(a_1(t), a_2(t))$ where for each integer $n \geqslant 0$

$$a_1(t) = \begin{cases} e^{-t} + \sin^2 t & \text{for} \quad 2n\pi \leqslant t \leqslant (2n+1)\pi \\ e^{-t} & \text{for} \quad (2n+1)\pi < t < 2(n+1)\pi, \end{cases}$$

and

$$a_2(t) = 2e^{-t} + \sin^2 t - a_1(t).$$

Then $\lambda_1[A(t)] = e^{-t}$, and $\lambda_2[A(t)] = e^{-t} + \sin^2 t$. Thus $\int_0^\infty t\lambda_1[A(t)]\, dt < \infty$, and $\int_0^\infty t\lambda_2[A(t)]\, dt = \infty$. Since $A(t)$ is a diagonal matrix the matrix equation (0.2) is just two scalar equations. As above using the scalar case of Corollary 1.5, we see that since $\int_a^\infty t a_k(t)\, dt = \infty$ for $k = 1, 2$, solutions to the scalar equations $x'' = a_k(t)\, x$ can have only zero limits, forcing all finite matrix limits to be zero.

However, under certain circumstances results in the direction indicated above are available. We first make an assumption concerning the asymptotic behavior of $\Lambda^* A(t)\, \Lambda$ for the following theorem and corollary.

THEOREM 2.3. *If $A(t) \geqslant 0$, if $X(t)$ is a matrix solution of the problem* (0.2) *and* (0.4), *if rank $\Lambda = r$, and if* $\lim_{t\to\infty} \Lambda^* A(t)\, \Lambda$ *exists as a finite matrix, then* $\lim_{t\to\infty} \lambda_r[A(t)] = 0$.

Proof. Consider $\Lambda^*\Lambda \geqslant 0$. Let $\{b_1, \ldots, b_n\}$ be orthonormal vectors such that $b_k^*\Lambda^*\Lambda b_k = \lambda_k(\Lambda^*\Lambda)$. Since rank $(\Lambda^*\Lambda) =$ rank $\Lambda = r$, we know that $\lambda_k(\Lambda^*\Lambda) = 0$ for $1 \leqslant k \leqslant n - r$, and $\lambda_k(\Lambda^*\Lambda) > 0$ for $n - r + 1 \leqslant k \leqslant n$. Let $\mathscr{V}_r$ be the r-dimensional vector space spanned by $\{b_{n-r+1}, \ldots, b_n\}$. Consider next the space spanned by the set $\{\Lambda\, b_{n-r+1}, \ldots, \Lambda\, b_n\}$. This space is r-dimensional since the spanning set is orthogonal. Now we show there is a vector e_r such that $e_r^*\Lambda^* A(t)\, \Lambda e_r \geqslant \lambda_r[A(t)]$ by the use of Rayleigh quotients (see [7, p. 111]). Letting $\mathscr{V}_n$ denote Euclidean n-space, we have the following.

$$\lambda_1[A(t)] = \min_{c\in\mathscr{V}_n} \frac{c^*A(t)c}{c^*c} \equiv c_1^*A(t)c_1 \leqslant \min_{e\in\mathscr{V}_r} \frac{e^*\Lambda^*A(t)\,\Lambda e}{e^*\Lambda^*\Lambda e} \equiv e_1^*\Lambda^*A(t)\,\Lambda e_1 .$$

$$\lambda_2[A(t)] = \min_{\substack{c\in\mathscr{V}_n \\ c^*c_1=0}} \frac{c^*A(t)c}{c^*c} \equiv c_2^*A(t)c_2 \leqslant \min_{\substack{e\in\mathscr{V}_r \\ e_1^*\Lambda^*\Lambda e=0}} \frac{e^*\Lambda A(t)e}{e^*\Lambda^*\Lambda e} \equiv e_2^*\Lambda^*A(t)\Lambda e_2 .$$

$$\vdots$$

$$\lambda_r[A(t)] = \min_{\substack{c\in\mathscr{V}_n \\ c^*c_k=0 \\ 1\leqslant k\leqslant r-1}} \frac{c^*A(t)c}{c^*c} \equiv c_r^*A(t)c_r$$

$$\leqslant \min_{\substack{e\in\mathscr{V}_r \\ e_k^*\Lambda^*\Lambda e=0 \\ 1\leqslant k\leqslant r-1}} \frac{e^*\Lambda^*A(t)\,\Lambda e}{e^*\Lambda^*\Lambda e} \equiv e_r^*\Lambda^*A(t)\,\Lambda e_r .$$

Now define $x(t) = e_r^* X^*(t)\, X(t)\, e_r$. Then $x(\infty) = e_r^* \Lambda^* \Lambda e_r = 1$, and as in (2.3), $x''(t) \geqslant 0$. This forces $\liminf_{t\to\infty} x''(t) = 0$. Hence

$$\liminf 2e_r^* X^*(t)\, A(t)\, X(t)\, e_r = 0.$$

But the limit of this expression exists, and, hence, must be zero. Therefore,

$$\lim_{t\to\infty} e_r^* \Lambda^* A(t)\, \Lambda e_r = \lim_{t\to\infty} \lambda_r[A(t)] = 0.$$

COROLLARY 2.4. *If* $A(t) \geqslant 0$, *if* $X(t)$ *is a matrix solution to the problem* (0.2) *and* (0.4), *if* rank $\Lambda = r$, *and if* $A(\infty)$ *exists as a finite matrix, then* $\lambda_r[A(t)] \to 0$ *as* $t \to \infty$.

We now turn to a condition on $A(t)$ which allows a characterization of rank Λ in terms of the eigenvalues of $A(t)$. We restrict $A(t)$ to matrices which may be block diagonalized in a certain manner to be described below. The characterization is preceded by the following lemma.

LEMMA 2.5. *Let* $A(t)$ *be semidefinite. If the problem* (0.2) *and* (0.3) *has a solution* $X(t)$, *then the problem* (0.2) *and* (0.4) *has a unique solution.*

Proof. As in Lemma 1.1, we write a general solution as $Z(t) = X(t)C + t[I + G(t)]\, D$. Suppose $Z(\infty) = \Lambda$. Then we shall see that $D = 0$. First, as $t \to \infty$,

$$t[I + G(t)]\, D \to Z(\infty) - X(\infty)\, C = \Lambda - C \equiv M.$$

In terms of the elements of the matrices involved, this is

$$td_{ik} - m_{ik} + t \sum_{j=1}^{n} g_{ij}(t)\, d_{jk} \to 0, \quad \text{as} \quad t \to \infty.$$

But,

$$t^{-1} \Big[td_{ik} - m_{ik} + t \sum_j g_{ij}(t)\, d_{jk}\Big] \to 0, \quad \text{as} \quad t \to \infty,$$

also. Hence,

$$-\sum_j g_{ij}(t)\, d_{jk} \to d_{ik}\,, \quad \text{as} \quad t \to \infty,$$

but we know that $\sum_j g_{ij}(t)\, d_{jk} \to 0$. So, $d_{ik} = 0$, and $D = 0$. Therefore $Z(t) = XC$, forcing $C = \Lambda$, since otherwise $Z(\infty) \neq \Lambda$.

THEOREM 2.6. *Let* $A(t) \geqslant 0$. *Let* Q *be a constant invertible matrix such that*

$$Q^{-1}A(t)Q = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix},$$

where $A_1(t)$ *is* $r \times r$, $A_2(t)$ *is* $(n-r) \times (n-r)$, *and the zeros are of appropriate size for this block diagonal form. Suppose that the eigenvalues of* $A(t)$ *are distributed so that* $\lambda_r[A_1(t)] = \lambda_r[A(t)]$, *and* $\lambda_1[A_2(t)] = \lambda_{r+1}[A(t)]$. *Let* $X(t)$ *be a nonsingular solution to the problem* (0.2) *and* (0.4). *Then rank* $\Lambda = r$ *if and only if*

$$\int_a^\infty t\lambda_r[A(t)]\,dt < \infty \tag{2.6}$$

and

$$\int_a^\infty t\lambda_{r+1}[A(t)]\,dt = \infty. \tag{2.7}$$

Proof. Let $B(t) \equiv Q^{-1}A(t)Q$. Then for $Y(t) \equiv Q^{-1}X(t)Q$, we have $Y''(t) = B(t)\,Y(t)$ and $Y(\infty) = Q^{-1}\Lambda Q \equiv M$. Writing $Y(t)$ in block form as,

$$Y(t) = \begin{pmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{21}(t) & Y_{22}(t) \end{pmatrix},$$

where Y_{11} and Y_{22} are squares of dimension r and $n-r$, respectively, and Y_{12} and Y_{21} are blocks of appropriate dimension, we have $Y_{2k}''(t) = A_2(t)Y_{2k}(t)$ for $k = 1, 2$. But, by the $(n-r)$-dimensional case of Theorem 2.1 we see that $Y_{2k}(t) \to 0$ as $t \to \infty$. Note, since every square matrix solution with limit goes to zero, we have that it is also true for vector solutions, and, hence, for nonsquare matrix solutions. Thus,

$$M = Q^{-1}\Lambda Q = \begin{pmatrix} M_{11} & M_{12} \\ 0 & 0 \end{pmatrix},$$

where the dimensions are as before. Therefore, rank $\Lambda = \operatorname{rank} Q^{-1}\Lambda Q \leqslant r$.

Now to see that rank $\Lambda = r$, we assume that rank $\Lambda = s < r$ and show that this prevents $X(t)$ from being nonsingular. Now, rank $M = s$, and by rearranging the columns of $Y(t)$ and, hence of M, we may have s independent columns of M first and the rest of the columns of M as a linear combination of the first s. Thus, by letting m_j denote the jth column of M, we have for $s+1 \leqslant j \leqslant n$,

$$m_j = \sum_{k=1}^{s} \alpha_{jk} m_k \,. \tag{2.8}$$

Define an r-vector $\bar{z}_j$ for $s+1 \leqslant j \leqslant n$ in the following manner,

$$\bar{z}_j(t) = \bar{y}_j(t) - \sum_{k=1}^{s} \alpha_{jk}\bar{y}_k(t), \tag{2.9}$$

where $\bar{y}_k$ is the first r elements of y_k, the kth column of Y. Hence, $\bar{z}_j(t) \to 0$ as $t \to \infty$. Now, $\bar{z}_j''(t) = A_1(t)\,\bar{z}_j(t)$. Therefore, by Corollary 1.5 and Lemma 2.5 in the r-dimensional case, there is on some interval $[a_0, \infty)$ a solution $Z(t)$ of the problem (0.2) and (0.3) with A replaced by A_1, and every solution with finite limit may be written as $Z(t)\,C$. In particular, every such vector solution may be written as $\bar{z}_j(t) = Z(t)\,\bar{c}_j$. By (2.8) and (2.9) we see that $\bar{z}_j(\infty) = 0$, and, hence, that $\bar{c}_j = 0$. Therefore, $\bar{z}_j(t) \equiv 0$. Hence, the $r \times n$ submatrix $(Y_{11}(t)\ Y_{12}(t))$ of $Y(t)$ has rank at most s. But, $(Y_{21}(t)\ Y_{22}(t))$ has rank at most $n - r$, and so $Y(t)$ has rank at most $s + n - r < n$. This means that $Y(t)$ is singular, whether or not its columns have been rearranged. Consequently, $X(t)$ must also be singular, contradicting the hypothesis. Therefore, rank $\Lambda = r$.

Conversely, if

$$\int_a^\infty t\lambda_r[A(t)]\,dt = \infty,$$

then rank $\Lambda < r$, or if

$$\int_a^\infty t\lambda_{r+1}[A(t)]\,dt < \infty,$$

then rank $\Lambda > r$ both of which are contradictory.

It should be pointed out that nonsingular solutions exist. In fact, Wintner [3] has shown that for any nonsingular matrix B there is a unique bounded nonsingular solution of (0.2) on $[a, \infty)$ satisfying $X(a) = B$ whenever $A(t) \geqslant 0$.

The following example shows a matrix $A(t)$ which may be put in the block diagonal form of Theorem 2.6.

Example 2.7. *Let* $A(t) = (a_{ij}(t))$, a 2×2 matrix be defined on $[1, \infty)$ as follows:

$$a_{11}(t) = a_{22}(t) = 2(t+1)(2t+1)^{-1}\,t^{-2},$$

and

$$a_{12}(t) = a_{21}(t) = -2t^{-1}(2t+1)^{-1}$$

for each $t \geqslant 1$. Then

$$\lambda_1[A(t)] = 2t^{-2}(2t+1)^{-1},$$

and

$$\lambda_2[A(t)] = 2t^{-2}.$$

Hence

$$\int_1^{\infty} t\lambda_1[A(t)]\, dt < \infty,$$

and

$$\int_1^{\infty} t\lambda_2[A(t)]\, dt = \infty.$$

Thus, we know by Corollary 1.5 that the problem (0.2) and (0.3) has no solution. As indicated above, this matrix $A(t)$ can be diagonalized on $[1, \infty)$ by a constant matrix transformation $Q^*A(t)\,Q$ where

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

A nonsingular solution for this $A(t)$ is given by $X(t) = (x_{ij}(t))$ where $x_{11}(t) = x_{22}(t) = (t+1)\,t^{-1}$ and $x_{12}(t) = x_{21}(t) \equiv 1$ on $[1, \infty)$. Thus

$$\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

which has rank $\Lambda = 1$.

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